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2007 J. Phys. A: Math. Theor. 40 15083

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# Collapse models with non-white noises

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Received 29 August 2007, in final form 19 October 2007

Published 28 November 2007

Online at [stacks.iop.org/JPhysA/40/15083](http://stacks.iop.org/JPhysA/40/15083)

## Abstract

We set up a general formalism for models of spontaneous wavefunction collapse with dynamics represented by a stochastic differential equation driven by general Gaussian noises, not necessarily white in time. In particular, we show that the non-Schrödinger terms of the equation induce the collapse of the wavefunction to one of the common eigenstates of the collapsing operators, and that the collapse occurs with the correct quantum probabilities. We also develop a perturbation expansion of the solution of the equation with respect to the parameter which sets the strength of the collapse process; such an approximation allows one to compute the leading-order terms for the deviations of the predictions of collapse models with respect to those of standard quantum mechanics. This analysis shows that to leading order, the ‘imaginary noise’ trick can be used for non-white Gaussian noise.

PACS numbers: 03.65.Ca, 03.65.Yz

## 1. Introduction

Models of spontaneous wavefunction collapse [1] provide a simple consistent resolution to the measurement problem of quantum mechanics [2], and at the same time provide precise indications for experiments which are more likely to detect possible violations of quantum linearity [3]: these include e.g. fullerene diffraction experiments, decay of supercurrents, excitation of bound atomic and nuclear systems and several cosmological observations. The dynamics is represented by a stochastic Schrödinger equation of the form

$$d|\psi_t\rangle = \left[ -\frac{i}{\hbar} H dt + \sqrt{\gamma} \sum_{i=1}^N (A_i - \langle A_i \rangle_t) dW_{i,t} - \frac{\gamma}{2} \sum_{i=1}^N (A_i - \langle A_i \rangle_t)^2 dt \right] |\psi_t\rangle, \quad (1)$$

where  $H$  is the standard quantum Hamiltonian of the system,  $A_i$  are a set of commuting self-adjoint operators to whose eigenstates the wavefunction is driven during the collapse process,

$W_{i,t}$  are  $N$  independent standard Wiener processes defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , the average  $\langle A_i \rangle_t \equiv \langle \psi_t | A_i | \psi_t \rangle$  is the standard quantum expectation of  $A_i$ , and  $\gamma$  is a positive constant which sets the strength of the collapse process.

The several collapse models which have been so far proposed differ from each other basically only by the choice of the localizing operators: in GRW-type models [4], the set  $\{A_i\}$  corresponds to the set of position operators of the constituents of the given physical system, or some function of them; dissipative effects can be included by taking  $A_i$  to be a function of both the position and the momentum operator of a particle [5], the resulting operator being non-Hermitian; in the CSL model for identical particles [6], the index  $i$  is replaced by the space coordinate  $\mathbf{x}$  and  $A(\mathbf{x})$  becomes a function of the density number operator  $a^\dagger(\mathbf{x})a(\mathbf{x})$ ; in energy driven models [7] there appears only one operator  $A$ , which is identified with the Hamiltonian  $H$ . Finally, reduction models related to gravitational effects [8] can also be cast in the form (1), as shown in [9].

The reduction properties of equation (1) can easily be verified by computing the time evolution of the variance  $V_A(t) \equiv \langle A^2 \rangle_t - \langle A \rangle_t^2$ , of an operator  $A$  which commutes with all the operators  $A_i$ ; as shown e.g. in [7], by using standard Itô calculus rules, and by setting  $H = 0$ , one gets for the average value  $\mathbb{E}_{\mathbb{P}}[V_A(t)]$  the following equation:

$$\mathbb{E}_{\mathbb{P}}[V_A(t)] = V_A(0) - 4\gamma \sum_{i=1}^N \int_0^t ds \mathbb{E}_{\mathbb{P}}[C_{A,A_i}^2(s)], \quad (2)$$

with  $C_{A,A_i}(t) \equiv \langle (A - \langle A \rangle_t)(A_i - \langle A_i \rangle_t) \rangle_t$ . Since the integrand on the right-hand side is a non-negative quantity, the above relation, when applied to any operator  $A_i$ , implies that, for large times, the variance  $V_{A_i}(t)$  converges to 0 for any realization of the noise, with the possible exception only of a subset of  $\Omega$  of measure 0; this means that any initial state  $|\psi_0\rangle$  converges asymptotically, with probability 1, to one of the common eigenstates of the operators  $A_i$ . When  $H \neq 0$  and moreover it does not commute with the other operators  $A_i$ , equation (1) induces only an approximate collapse, the degree of approximation depending on the relative strength of the Schrödinger term and of the collapse terms which define the equation.

A very useful mathematical property of equation (1) is that its physical predictions concerning the outcomes of measurements are, in terms of statistical expectations, invariant under a phase change in the noise. As a matter of fact, let us consider the following class of stochastic Schrödinger equations:

$$d|\psi_t\rangle = \left[ -\frac{i}{\hbar} H dt + \sqrt{\gamma} \sum_{i=1}^N (\xi A_i - \xi_R \langle A_i \rangle_t) dW_{i,t} - \frac{\gamma}{2} \sum_{i=1}^N (|\xi|^2 A_i^2 - 2\xi \xi_R A_i \langle A_i \rangle_t + \xi_R^2 \langle A_i \rangle_t^2) dt \right] |\psi_t\rangle, \quad (3)$$

where  $\xi = \xi_R + i\xi_I$  is a constant complex factor; of course, when  $\xi = 1$  we recover our original collapse equation. An easy application of Itô calculus leads to the following equation for the density matrix  $\rho(t) = \mathbb{E}_{\mathbb{P}}[|\psi_t\rangle\langle\psi_t|]$ :

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H, \rho(t)] + \frac{\gamma}{2}|\xi|^2 \sum_{i=1}^N (2A_i \rho(t) A_i - \{A_i^2, \rho(t)\}), \quad (4)$$

which is of the Lindblad type and has the remarkable property that it depends only on the square modulus of  $\xi$ . Since, within collapse models, the statistics of the outcome of experiments [10] can be expressed by the averages  $\mathbb{E}_{\mathbb{P}}[\langle \psi_t | O | \psi_t \rangle] \equiv \text{Tr}[\rho(t) O]$ , where  $O$  is a self-adjoint operator, we see that in order to compute experimental predictions, one can use in place of

equation (1) any stochastic equation of the type (3) which satisfies the constraint  $|\xi| = 1$ ; in most cases, it is convenient to choose the equation corresponding to  $\xi = i$ , since it is linear, thus much easier to solve [11]. Of course, this does not mean that all equations of the form (3), having the same value of  $|\xi|$ , are equivalent; in contrast, in general they generate completely different evolutions for the wavefunction. For example, when  $\xi = 1$ , as we have seen, the corresponding equation induces collapse of the wavefunction, since the property (2) for the variance holds true; while on the other hand with  $\xi = i$ , the corresponding equation is linear; thus, the wavefunction does not collapse in this case. This notwithstanding, all quantities of the form  $\mathbb{E}_{\mathbb{P}}[\langle \psi_t | O | \psi_t \rangle]$  turn out to be the same for the two equations, and for similar ones with complex  $\xi$  of modulus unity.

The aim of this paper is to generalize equation (1) in order to include also non-white, Gaussian, stochastic processes. Some results have already appeared in the literature [12–15], but a general analysis is still lacking, mainly because the new dynamics is not Markovian and thus is more difficult to describe mathematically. There are two main reasons why one should consider collapse models driven by non-white noises. First of all, it is important to understand how the collapse mechanism and other physical properties, such as the time evolution of the mean energy, depend on the type of noise driving the collapse of the wavefunction. Since these properties are directly connected to physical predictions which differ from those given by standard quantum mechanics, such differences, and thus the possibility of experimentally testing collapse models, could significantly change depending on the type of noise entering the collapse equation. The second reason for such an analysis is that a non-white noise, unlike a Wiener process, can be identified with a physical field; accordingly, one can try to connect the collapse mechanism to some other physical process occurring in Nature, possibly having a cosmological origin; we will come back to this point in the final section.

This paper contains two main sections, which set up the general formalism for non-Markovian collapse models by following two different paths: in section 2, we follow the same argument used in [6] to derive, from a generic diffusion process in Hilbert space, equation (1) as the correct collapse equation; in section 3 instead we follow the strategy used in [16] to obtain the same equation from general requirements on the dynamics of the density matrix. We will prove the two following main results.

- (1) We will show that, if one neglects the quantum Hamiltonian  $H$ , the dynamics leads to the collapse of the wavefunction to one of the common eigenstates of the localizing operators, with the correct quantum probabilities.
- (2) We will develop a perturbation expansion of the solution of the equation with respect to the coupling constant  $\sqrt{\gamma}$ ; when applied to experimental predictions on microscopic systems, it provides the leading-order term for the deviation of such predictions from those given by standard quantum mechanics.

Concerning this second result, we will also show that, at least to order  $\gamma$ , the equation for the statistical operator depends only on the absolute value of  $\xi$ , precisely as discussed before for the white-noise case; this means that, to this order, one can employ the useful trick of replacing the real-noise ( $\xi = 1$ ), nonlinear collapsing equation with an imaginary-noise ( $\xi = i$ ), linear non-collapsing equation, thus considerably simplifying calculations.

Throughout the calculations, one has to make sure that, at any stage, one recovers the correct white-noise limit; however, one has to keep in mind that the white-noise limit of the non-white collapsing equation will not be given by (1) or (3), since these are Itô equations; instead, as explained e.g. in [17], an equation containing non-white noises reduces, in the white-noise limit, to a Stratonovich equation. Accordingly, we write the Stratonovich equation

corresponding to the Itô equation (3), which is [6]

$$\begin{aligned} \frac{d|\psi_t\rangle}{dt} &= \left[ -\frac{i}{\hbar}H + \sqrt{\gamma} \sum_{i=1}^N (\xi A_i - \xi_R \langle A_i \rangle_t) w_i(t) \right. \\ &\quad \left. - \gamma \xi_R \sum_{i=1}^N (\xi A_i^2 - 2\xi A_i \langle A_i \rangle_t - \xi_R \langle A_i^2 \rangle_t + 2\xi_R \langle A_i \rangle_t^2) \right] |\psi_t\rangle \\ &= \left[ -\frac{i}{\hbar}H + \sqrt{\gamma} \sum_{i=1}^N (\xi A_i - \xi_R \langle A_i \rangle_t) (w_i(t) + 2\sqrt{\gamma} \xi_R \langle A_i \rangle_t) \right. \\ &\quad \left. - \gamma \xi_R \sum_{i=1}^N (\xi A_i^2 - \xi_R \langle A_i^2 \rangle_t) \right] |\psi_t\rangle. \end{aligned} \quad (5)$$

In the second expression, we have written the third term in square brackets in a form that corresponds to the white-noise limit of equation (18), with the remainder of the order  $\gamma$  part playing the role of a shift in the mean value of the noise, corresponding to a change of measure, as discussed in section 4. Of course, whether one uses equation (3) or (5), the corresponding equation for the statistical operator  $\rho(t)$ , which determines the evolution of statistical ensembles of states, will always be given by (4).

## 2. Linear and nonlinear collapse equations

Following the path outlined in [6] to construct the continuous generalization of the GRW model in terms of an Itô stochastic differential equation of type (1), let us consider a diffusion process for the wavefunction in Hilbert space having the form

$$\frac{d|\phi(t)\rangle}{dt} = \left[ -\frac{i}{\hbar}H + \sqrt{\gamma} \xi \sum_{i=1}^N A_i w_i(t) + O \right] |\phi(t)\rangle, \quad (6)$$

where, as before,  $H$  is the standard quantum Hamiltonian of the system,  $A_i$  are commuting self-adjoint operators,  $\gamma$  is a positive coupling constant,  $\xi = \xi_R + i\xi_I$  is a constant complex factor, while  $O$  is a linear operator yet to be defined<sup>5</sup>. The noises  $w_i(t)$  are real Gaussian random processes defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{Q})$  whose mean and correlation functions are, respectively,

$$\mathbb{E}_{\mathbb{Q}}[w_i(t)] = 0, \quad \mathbb{E}_{\mathbb{Q}}[w_i(t_1)w_j(t_2)] = D_{ij}(t_1, t_2). \quad (7)$$

When  $\xi_R \neq 0$ , which is the case for collapse models, equation (6) does not preserve the norm of the wavefunction; therefore, we introduce the normalized vector:

$$|\psi(t)\rangle = \frac{|\phi(t)\rangle}{\| |\phi(t)\rangle \|}, \quad (8)$$

(assuming of course that the norm of  $|\phi(t)\rangle$  does not vanish) which we take as the physical vector describing the random state of the system at time  $t$ .

The measure  $\mathbb{Q}$  previously introduced is not the correct physical probability since it does not lead to a collapse that respects the Born probability rule; the right physical probability, which we shall call  $\mathbb{P}$ , is defined as follows:

$$\mathbb{P}[F] = \mathbb{E}_{\mathbb{Q}}[1_F \langle \phi(t) | \phi(t) \rangle] \quad \forall F \in \mathcal{F}, \quad (9)$$

<sup>5</sup> As we shall see, the operator  $O$  will not be a standard linear operator, since it will act on a vector also through its dependence on the noises  $w_i(t)$ , by means of functional derivatives. In this respect, our approach differs from the standard one based on Itô's diffusion equations in Hilbert spaces.

where  $1_F$  is the indicator function associated with the measurable subset  $F$  of  $\Omega$ . This definition corresponds to the assumption of the GRW model, according to which a collapse (in space) is more likely to occur where the wavefunction is larger, as postulated by the Born probability rule.

To summarize, an initial state  $|\psi(0)\rangle$  is driven by the stochastic dynamics into an ensemble of states  $|\psi(t)\rangle$  of the form (8), where  $|\phi(t)\rangle$  solves equation (6); the distribution of states within the ensemble is given by the probability  $\mathbb{P}$  defined in (9).

Note that the definition (9) of  $\mathbb{P}$  not only matches with the Born probability rule, but is also necessary in order to prevent the possibility of using the collapse mechanism to send information at a speed faster than the speed of light. As shown in [18], when the dynamics of a statistical operator  $\rho(t)$  is nonlinear, it is in general possible to send faster-than-light signals which can be used by two spacelike separated observers to communicate with each other; this possibility is instead forbidden when the evolution is linear. In our case, according to the previous assumptions,  $\rho(t)$  is defined as follows:

$$\rho(t) = \mathbb{E}_{\mathbb{P}}[|\psi(t)\rangle\langle\psi(t)|]; \quad (10)$$

but according to equations (8) and (9) we have the mathematical equality:

$$\mathbb{E}_{\mathbb{P}}[|\psi(t)\rangle\langle\psi(t)|] = \mathbb{E}_{\mathbb{Q}}[|\phi(t)\rangle\langle\phi(t)|], \quad (11)$$

and since  $|\phi(t)\rangle$  solves the linear equation (6), it follows that the operator which maps  $\rho(t)$  to  $\rho(t+dt)$  is linear, thus not allowing faster-than-light signaling of the type just discussed.

We still need to prove that equation (9) correctly defines a probability measure; it is easy to check that all properties are satisfied except for the normalization condition  $\mathbb{P}[\Omega] = \mathbb{E}_{\mathbb{Q}}[\langle\phi_t|\phi_t\rangle] = 1$ , which in general is not fulfilled unless the operator  $O$  takes a particular form. In fact, by using the Furutsu–Novikov formula [19]:

$$\mathbb{E}_{\mathbb{Q}}[F[\{w(t)\}]w_i(t)] = \sum_{j=1}^N \int_0^t ds D_{ij}(t,s) \mathbb{E}_{\mathbb{Q}} \left[ \frac{\delta F[\{w(t)\}]}{\delta w_j(s)} \right] \quad (12)$$

which holds for a generic functional  $F[\{w(t)\}]$  of the Gaussian noises  $w_i(t)$ ,  $i = 1, \dots, N$  satisfying (7), and computed from initial time 0 to time  $t$ , one can immediately prove that

$$\frac{d}{dt} \mathbb{E}_{\mathbb{Q}}[\langle\phi_t|\phi_t\rangle] = 0 \quad \text{if} \quad O = -2\sqrt{\gamma}\xi_{\mathbb{R}} \sum_{i,j=1}^N A_i \int_0^t ds D_{ij}(t,s) \frac{\delta}{\delta w_j(s)}. \quad (13)$$

Accordingly, the linear equation which, together with equation (9), induces collapse of the wavefunction with the correct quantum probabilities and, at the same time, does not allow one to use the collapse process to send signals at faster-than-light speed, is [14]

$$\frac{d|\phi(t)\rangle}{dt} = \left[ -\frac{i}{\hbar}H + \sqrt{\gamma}\xi \sum_{i=1}^N A_i w_i(t) - 2\sqrt{\gamma}\xi_{\mathbb{R}} \sum_{i,j=1}^N A_i \int_0^t ds D_{ij}(t,s) \frac{\delta}{\delta w_j(s)} \right] |\phi(t)\rangle. \quad (14)$$

As foreseen, this equation is non-Markovian and for this reason is highly non-trivial, since the future evolution, which involves the whole past, depends on the combined effect of the standard Hamiltonian  $H$  and the collapsing operators  $A_i$ . This dynamics is not easy to unfold if these operators do not commute among themselves, as is usually the case.

The equation for the normalized vector  $|\psi(t)\rangle$  does not have a closed form, unless the functional derivative of  $|\phi(t)\rangle$  can be explicitly computed; as we shall see in the next sections, this happens when the Hamiltonian  $H$  is neglected (or when it commutes with the operators  $A_i$ ),

and when one writes the evolution as a perturbation expansion with respect to the relevant parameters.

Before moving on, we make a few comments about the change of measure as defined in (9). In the white-noise case one can prove, under suitable hypotheses on the operators  $H$  and  $A_i$  (see e.g. [20]), that  $\langle \phi(t) | \phi(t) \rangle$  is martingale with  $\mathbb{Q}$ -mean equal to 1; this ensures that one can consistently use  $\langle \phi(t) | \phi(t) \rangle$  as a Radon–Nikodym derivative of a new probability measure  $\mathbb{P}$  with respect to  $\mathbb{Q}$ , as we have assumed more heuristically in the previous paragraphs. Here we will not attempt to prove that  $\langle \phi(t) | \phi(t) \rangle$  satisfies the required properties also in the more general case of non-white Gaussian processes, leaving the analysis of the conditions under which this is true to future research. Secondly, Girsanov’s theorem provides, in the white-noise case, a connection between Wiener processes with respect to the measure  $\mathbb{Q}$  and Wiener processes with respect to the transformed measure  $\mathbb{P}$ . It would be interesting to see whether a similar theorem can be proved also in the non-white-noise case, and whether  $\mathbb{Q}$ -Gaussian processes can be connected to  $\mathbb{P}$ -Gaussian processes; we will come back on this point in section 4.

### 2.1. Collapse of the state vector

We now show that, when the standard quantum Hamiltonian  $H$  is set to 0, the dynamics induces the collapse of the state vector  $|\psi(t)\rangle$  to one of the common eigenstates of the operators  $A_i$ . As shown in [14], if  $H$  is neglected so that all operators entering the equation commute among themselves, then the functional derivative can be explicitly computed and equation (14) reduces to

$$\frac{d|\phi(t)\rangle}{dt} = \left[ \sqrt{\gamma} \xi \sum_{i=1}^N A_i w_i(t) - 2\gamma \xi \xi_R \sum_{i,j=1}^N A_i A_j F_{ij}(t) \right] |\phi(t)\rangle, \quad (15)$$

where we have defined

$$F_{ij}(t) = \int_0^t ds D_{ij}(t, s). \quad (16)$$

That equation (15) is equivalent to equation (14) in the limit  $H = 0$  can easily be seen by integration of equation (15), which is trivial since all operators commute, from which one obtains the relation:

$$\frac{\delta}{\delta w_j(s)} |\phi(t)\rangle = \sqrt{\gamma} \xi A_j |\phi(t)\rangle, \quad s \leq t, \quad (17)$$

a relation which will often be used in the following calculations.

The equation for the normalized vector  $|\psi(t)\rangle$  can now be directly computed from the definition (8):

$$\frac{d|\psi(t)\rangle}{dt} = \left[ \sqrt{\gamma} \sum_{i=1}^N (\xi A_i - \xi_R \langle A_i \rangle_t) w_i(t) - 2\gamma \xi_R \sum_{i,j=1}^N (\xi A_i A_j - \xi_R \langle A_i A_j \rangle_t) F_{ij}(t) \right] |\psi(t)\rangle, \quad (18)$$

with the expectations  $\langle \dots \rangle_t$  computed in the state  $|\psi(t)\rangle$ , that is  $\langle O \rangle_t \equiv \langle \psi(t) | O | \psi(t) \rangle$ .

The equation for the statistical operator  $\rho(t)$  can now be computed either from equation (18) through the definition of equation (10) or from equation (15) through the

equivalence of equation (11); in both cases one gets

$$\frac{d}{dt}\rho(t) = \gamma|\xi|^2 \sum_{i,j=1}^N (A_i\rho(t)A_j + A_j\rho(t)A_i - A_iA_j\rho(t) - \rho(t)A_jA_i)F_{ij}(t), \quad (19)$$

which correctly reduces to (4) in the white-noise limit<sup>6</sup>.

We have now all the necessary formulas to compute the time evolution of quantities such as  $\mathbb{E}_{\mathbb{P}}[\langle A^n \rangle_t]$  and  $\mathbb{E}_{\mathbb{P}}[\langle A \rangle_t^n]$ , where  $A$  is a self-adjoint operator commuting with all the operators  $A_i$ . Because of the relation  $\mathbb{E}_{\mathbb{P}}[\langle A^n \rangle_t] = \text{Tr}[A^n \rho(t)]$ , which is a consequence of equation (10), and because of the trace-preserving structure of equation (19), one immediately has

$$\frac{d}{dt}\mathbb{E}_{\mathbb{P}}[\langle A^n \rangle_t] = 0. \quad (20)$$

By using the change of measure (9) and through a direct calculation, one finds for  $\mathbb{E}_{\mathbb{P}}[\langle A \rangle_t^n]$  instead that

$$\begin{aligned} \frac{d}{dt}\mathbb{E}_{\mathbb{P}}[\langle A \rangle_t^n] &= 2\sqrt{\gamma}\xi_{\mathbb{R}}\mathbb{E}_{\mathbb{P}}[\langle A \rangle_t^{n-1}(n\langle AX \rangle_t - (n-1)\langle A \rangle_t\langle X \rangle_t)], \\ X &= \sum_{i=1}^N A_i w_i(t) - 2\sqrt{\gamma}\xi_{\mathbb{R}} \sum_{i,j=1}^N A_i A_j F_{ij}(t). \end{aligned} \quad (21)$$

The terms proportional to  $w_i(t)$  can be rewritten by using the Furutsu–Novikov formula, together with the equality:

$$\frac{\delta\langle O \rangle_t}{\delta w_j(s)} = \sqrt{\gamma}[\xi^* \langle A_j O \rangle_t + \xi \langle O A_j \rangle_t - 2\xi_{\mathbb{R}} \langle O \rangle_t \langle A_j \rangle_t], \quad (22)$$

which is valid for any operator  $O$ , and which can be directly proved from the definition  $\langle O \rangle_t \equiv \langle \phi(t) | O | \phi(t) \rangle / \langle \phi(t) | \phi(t) \rangle$ , together with equation (17). After a rather lengthy calculation, one can prove that equation (21) simplifies to

$$\frac{d}{dt}\mathbb{E}_{\mathbb{P}}[\langle A \rangle_t^n] = 4n(n-1)\gamma\xi_{\mathbb{R}}^2 \sum_{i,j} \mathbb{E}_{\mathbb{P}}[\langle A \rangle_t^{n-2} \langle (A_i - \langle A_i \rangle_t) A \rangle_t \langle (A_j - \langle A_j \rangle_t) A \rangle_t] F_{ij}(t). \quad (23)$$

We now apply equations (20) and (23) to compute the time evolution of the variance  $V_A(t) = \langle A^2 \rangle_t - \langle A \rangle_t^2$  of the operator  $A$ ; we obtain

$$\mathbb{E}_{\mathbb{P}}[V_A(t)] = V_A(0) - 8\xi_{\mathbb{R}}^2\gamma \sum_{i,j} \int_0^t ds \mathbb{E}_{\mathbb{P}}[\langle (A_i - \langle A_i \rangle_s) A \rangle_s \langle (A_j - \langle A_j \rangle_s) A \rangle_s] F_{ij}(s). \quad (24)$$

Now, the same argument used in equation (2) to prove the reduction of equation (1) holds true: when the matrix  $F_{ij}(s)$  is positive definite in the limit as  $t \rightarrow \infty$ , equation (24) is consistent if only if, for large times,  $\langle (A_i - \langle A_i \rangle_t) A \rangle_t$  goes to zero for any  $i$  almost surely (a.s.) (i.e., except on a subset of  $\Omega$  of realizations of the noise of  $\mathbb{P}$ -measure 0); in particular, if we take  $A$  equal to any one of the operators  $A_i$  we have

$$\lim_{t \rightarrow \infty} [\langle A_i^2 \rangle_t - \langle A_i \rangle_t^2] = \lim_{t \rightarrow \infty} V_{A_i}(t) = 0 \quad \text{a.s.} \quad \forall i, \quad (25)$$

<sup>6</sup> Note that in the white-noise limit, through equation (16) one encounters the integral of a delta function at the endpoint of an interval, which is  $\int_0^t ds \delta(t-s) = 1/2$ , since  $\delta(t) = \delta(-t)$  and  $\int_{-\infty}^{\infty} dt \delta(t) = 1$ . This enters both in comparing the white-noise limit of equation (19) to equation (4), and the white-noise limit of equation (18) to equation (5).



which is the desired result. Moreover, due to equation (20), the average value of  $\langle P_{a_i} \rangle_t$  remains constant in time, with  $P_{a_i}$  the projector on any eigenspace of  $A_i$  with eigenvalue  $a_i$ , which means that the collapse occurs with the correct quantum probabilities.

## 2.2. Perturbation expansion to order $\gamma$

The approximation used in the previous subsection, which consisted in neglecting the quantum Hamiltonian  $H$ , is useful when the system under study is macroscopic, since in this case the effect of the collapsing terms is typically much stronger than that of  $H$ : this is precisely the reason why collapse models ensure the localization of the wavefunction at the macroscopic level. For microscopic systems, on the other hand, such an approximation is no longer valid; just the reverse from the macroscopic case, at the microscopic level the effect of the collapsing terms represents typically only a small perturbation on the standard quantum evolution: for this reason collapse models agree very well with standard quantum-mechanical predictions. It then becomes meaningful, for micro-systems, to perform a perturbation expansion of the evolution of the state vector with respect to the parameter  $\sqrt{\gamma}$ , in order to compute the leading terms representing the deviations of the predictions of collapse models from those given by standard quantum mechanics. To this end, let us introduce the interaction picture operators and states:

$$A_i(t) = U^\dagger(t)A_iU(t), \quad |\phi^I(t)\rangle = U^\dagger(t)|\phi(t)\rangle, \quad U(t) = \exp\left(-\frac{i}{\hbar}Ht\right); \quad (26)$$

equation (14) then becomes

$$\frac{d|\phi^I(t)\rangle}{dt} = \left[ \sqrt{\gamma}\xi \sum_{i=1}^N A_i(t)w_i(t) - 2\sqrt{\gamma}\xi_R \sum_{i,j=1}^N A_i(t) \int_0^t ds D_{ij}(t,s) \frac{\delta}{\delta w_j(s)} \right] |\phi^I(t)\rangle. \quad (27)$$

The perturbation expansion of  $|\phi^I(t)\rangle$  with respect to the parameter  $\sqrt{\gamma}$  reads

$$|\phi^I(t)\rangle = |\phi_0^I(t)\rangle + \sqrt{\gamma}|\phi_1^I(t)\rangle + \gamma|\phi_2^I(t)\rangle + \dots, \quad (28)$$

while the functional derivative acts on  $|\phi^I(t)\rangle$  as follows:

$$\frac{\delta}{\delta w_j(s)}|\phi^I(t)\rangle = \sqrt{\gamma}\frac{\delta}{\delta w_j(s)}|\phi_1^I(t)\rangle + \gamma\frac{\delta}{\delta w_j(s)}|\phi_2^I(t)\rangle + \dots; \quad (29)$$

the second term must be of order  $\gamma$ , since the functional derivative brings down a term proportional to  $\sqrt{\gamma}$ ; for the same reason, the third term is of order  $\gamma^{3/2}$ . This means that the perturbation expansion can be explicitly carried out, despite the functional derivative appearing in equation (27), and we get the following results to order  $\gamma$ :

$$\text{order } 0 : \frac{d}{dt}|\phi_0^I(t)\rangle = 0, \quad (30)$$

$$\text{order } \sqrt{\gamma} : \frac{d}{dt}|\phi_1^I(t)\rangle = \xi \sum_{i=1}^N A_i(t)w_i(t)|\phi_0^I(t)\rangle, \quad (31)$$

$$\text{order } \gamma : \frac{d}{dt}|\phi_2^I(t)\rangle = \xi \sum_{i=1}^N A_i(t)w_i(t)|\phi_1^I(t)\rangle - 2\xi_R\xi \sum_{i,j=1}^N \int_0^t ds A_i(t)A_j(s)D_{ij}(t,s)|\phi_0^I(t)\rangle. \quad (32)$$

Going back to the Schrödinger picture, equation (14), to order  $\gamma$ , reduces to

$$\begin{aligned} \frac{d}{dt}|\phi(t)\rangle = & -\frac{i}{\hbar}H|\phi(t)\rangle + \left[ \sqrt{\gamma}\xi \sum_{i=1}^N A_i w_i(t) \right. \\ & \left. + \gamma\xi \sum_{i,j=1}^N \int_0^t ds A_i A_j(s-t)(\xi w_i(t)w_j(s) - 2\xi_R D_{ij}(t,s)) \right] |\phi_0(t)\rangle, \end{aligned} \quad (33)$$

with  $|\phi(t)\rangle$  the total wavefunction, and  $|\phi_0(t)\rangle$  its zeroth-order part. By use of equations (30)–(32), this can also be written entirely in terms of  $|\phi(t)\rangle$  as

$$\frac{d}{dt}|\phi(t)\rangle = \left[ -\frac{i}{\hbar}H + \sqrt{\gamma}\xi \sum_{i=1}^N A_i w_i(t) - 2\gamma\xi\xi_R \sum_{i,j=1}^N \int_0^t ds A_i A_j(s-t) D_{ij}(t,s) \right] |\phi(t)\rangle. \quad (34)$$

We can now see more clearly the non-Markovian nature of the evolution, because of the presence of the term  $A_j(s-t)$ , which depends on the past effect of  $H$  on  $A_j$ . When  $H = 0$ , one immediately sees that by using equation (16), the  $|\phi(t)\rangle$  evolution equation of equation (34) reduces to the evolution equation given in equation (15).

The corresponding equation for the normalized vector  $|\psi(t)\rangle$  defined by equation (8) can now be obtained by a straightforward application of the chain rule for differentiation, with the result

$$\begin{aligned} \frac{d}{dt}|\psi(t)\rangle = & \left[ -\frac{i}{\hbar}(H + i\hbar\gamma O_{ASA}) \right. \\ & \left. + \sqrt{\gamma} \sum_{i=1}^N (\xi A_i - \xi_R \langle A_i \rangle_t) w_i(t) + \gamma(O_{SA} - \langle O_{SA} \rangle_t) \right] |\psi(t)\rangle, \end{aligned} \quad (35)$$

again with the expectations  $\langle \dots \rangle_t$  computed in the state  $|\psi(t)\rangle$ . Here  $O_{ASA}$  and  $O_{SA}$  are respectively the anti-self-adjoint and self-adjoint parts of the operator  $O$  defined by the final term inside the square brackets of equation (34), and are given explicitly by

$$O_{ASA} = - \sum_{i,j=1}^N \int_0^t ds (\xi_R^2 [A_i, A_j(s-t)] + i\xi_I \xi_R \{A_i, A_j(s-t)\}) D_{ij}(t,s) \quad (36)$$

$$O_{SA} = - \sum_{i,j=1}^N \int_0^t ds (\xi_R^2 \{A_i, A_j(s-t)\} + i\xi_I \xi_R [A_i, A_j(s-t)]) D_{ij}(t,s). \quad (37)$$

The equation for the statistical operator can now be computed by resorting to relations (10) and (11). The calculation proceeds most directly from the  $|\phi(t)\rangle$  evolution equation given in equation (33), since in this equation all dependence on the noise is explicit. To the order to which we are working, we can replace the  $\gamma$  term in this equation by its expectation  $\mathbb{E}_{\mathbb{Q}}$ , giving the simplified evolution equation:

$$\begin{aligned} \frac{d}{dt}|\phi(t)\rangle = & -\frac{i}{\hbar}H|\phi(t)\rangle + \left[ \sqrt{\gamma}\xi \sum_{i=1}^N A_i w_i(t) \right. \\ & \left. - \gamma|\xi|^2 \sum_{i,j=1}^N \int_0^t ds A_i A_j(s-t) D_{ij}(t,s) \right] |\phi_0(t)\rangle. \end{aligned} \quad (38)$$

It is then straightforward to compute  $\mathbb{E}_{\mathbb{Q}}[|\phi(t)\rangle\langle\phi(t)|]$ , with the result<sup>7</sup>

$$\begin{aligned} \frac{d}{dt}\rho(t) = & -\frac{i}{\hbar}[H, \rho(t)] + |\xi|^2\gamma \sum_{i,j=1}^N \int_0^t ds D_{ij}(t, s) [A_i \rho(t) A_j(s-t) + A_j(s-t) \rho(t) A_i \\ & - A_i A_j(s-t) \rho(t) - \rho(t) A_j(s-t) A_i]. \end{aligned} \quad (39)$$

As we can see, the above equation, which is correct to order  $\gamma$ , depends only on the absolute value of  $\xi$ ; this means that all physical predictions are, at least to order  $\gamma$ , independent of the phase of  $\xi$ , precisely as in the white-noise case. As a consequence, in order to compute physical predictions, one can resort to equation (14) with  $\xi = i$ , which is much simpler since the integro-differential term vanishes, and one is left with a standard Schrödinger equation with a random Hermitian potential.

### 3. An alternative construction of the nonlinear collapse equation

We give in this section an alternative construction of the nonlinear collapse equation in the case of non-white Gaussian noise. Instead of starting from a linear equation and using a change of measure, we work throughout with a norm-preserving nonlinear equation, and use perturbation theory to determine the structure of the order  $\gamma$  term which guarantees that the corresponding density matrix evolution will have the Lindblad form [21] in the Markovian limit.

Thus, returning to equation (6), we now start from

$$\frac{d|\psi(t)\rangle}{dt} = \left[ -\frac{i}{\hbar}H + \sqrt{\gamma} \sum_{i=1}^N (\xi A_i - \xi_{\mathbb{R}} \langle A_i \rangle_t) w_i(t) + \gamma (B_{SA} - \langle B_{SA} \rangle_t) + \gamma B_{ASA} \right] |\psi(t)\rangle, \quad (40)$$

with  $B_{SA}$  and  $B_{ASA}$  respectively a self-adjoint operator and an anti-self-adjoint operator to be determined, and with  $w_i(t)$  now a non-white Gaussian noise obeying

$$\mathbb{E}_{\mathbb{R}}[w_i(t)] = 0, \quad \mathbb{E}_{\mathbb{R}}[w_i(t_1)w_j(t_2)] = D_{ij}(t_1, t_2) \quad (41)$$

with respect to the measure  $\mathbb{R}$ . We shall determine  $B_{SA}$  and  $B_{ASA}$  to simplify the evolution equation for  $\rho(t) = \mathbb{E}_{\mathbb{R}}[\hat{\rho}(t)] = \mathbb{E}_{\mathbb{R}}[|\psi(t)\rangle\langle\psi(t)|]$ , in such a way that in the Markovian limit it reduces to a Lindblad evolution in the first standard form,

$$\begin{aligned} \frac{d\rho(t)}{dt} = & \mathcal{L}\rho(t) \\ = & -\frac{i}{\hbar}[H, \rho(t)] + \sum_{i,j} a_{ij} \left( F_i \rho(t) F_j - \frac{1}{2} \{F_j^\dagger F_i, \rho(t)\} \right), \end{aligned} \quad (42)$$

with  $F_i$  suitable functions of  $\{A_i\}$ , and with the coefficients  $a_{ij}$  determined by the noise expectation  $D_{ij}$ .

By construction, equation (40) preserves the normalization of the state vector  $|\psi(t)\rangle$  under time evolution. From this equation, and its adjoint, one easily finds that the pure state density

<sup>7</sup> P Pearle has pointed out that, in the real-noise ( $\xi = 1$ ) case, this equation follows from differentiation of the integrated expression given in equation (4.11) of his *Physical Review* article cited in [12], which he suggests is exact when time-ordering is included.

matrix  $\hat{\rho}(t)$  obeys the evolution equation:

$$\begin{aligned} \frac{d\hat{\rho}(t)}{dt} = & -\frac{i}{\hbar}[H, \hat{\rho}(t)] + \left[ \sqrt{\gamma}i\xi_I \sum_{i=1}^N A_i w_i(t) + \gamma B_{\text{ASA}}, \hat{\rho}(t) \right] \\ & + \left\{ \sqrt{\gamma}\xi_R \sum_{i=1}^N (A_i - \langle A_i \rangle_t) w_i(t) + \gamma (B_{\text{SA}} - \langle B_{\text{SA}} \rangle_t), \hat{\rho}(t) \right\}. \end{aligned} \quad (43)$$

Taking the expectation of this, and retaining terms through order  $\gamma$  but dropping terms of order  $\gamma^{3/2}$ , we get

$$\begin{aligned} \frac{d\rho(t)}{dt} = & -\frac{i}{\hbar}[H, \rho(t)] + \sqrt{\gamma}i\xi_I \sum_{i=1}^N [A_i, \mathbb{E}_{\mathbb{R}}[\hat{\rho}(t)w_i(t)]] \\ & + \gamma[B_{\text{ASA}}, \rho(t)] + \sqrt{\gamma}\xi_R \sum_{i=1}^N \{A_i, \mathbb{E}_{\mathbb{R}}[\hat{\rho}(t)w_i(t)]\} \\ & - 2\sqrt{\gamma}\xi_R \sum_{i=1}^N \mathbb{E}_{\mathbb{R}}[\hat{\rho}(t)\langle A_i \rangle_t w_i(t)] + \gamma\{B_{\text{SA}} - \langle B_{\text{SA}} \rangle_t, \rho(t)\}. \end{aligned} \quad (44)$$

We now use the Furutsu–Novikov formula

$$\mathbb{E}_{\mathbb{R}}[F[\{w(t)\}]w_i(t)] = \sum_{j=1}^N \int_0^t ds D_{ij}(t, s) \mathbb{E}_{\mathbb{R}} \left[ \frac{\delta F[\{w(t)\}]}{\delta w_j(s)} \right], \quad (45)$$

first with  $F = \hat{\rho}(t)$ , and then with  $F = \langle A_i \rangle_t \hat{\rho}(t) = \hat{\rho}(t) \text{Tr} \hat{\rho}(t) A_i$ . By the chain rule,

$$\frac{\delta \hat{\rho}(t) \text{Tr} \hat{\rho}(t) A_i}{\delta w_j(s)} = \frac{\delta \hat{\rho}(t)}{\delta w_j(s)} \text{Tr} \hat{\rho}(t) A_i + \hat{\rho}(t) \text{Tr} \frac{\delta \hat{\rho}(t)}{\delta w_j(s)} A_i, \quad (46)$$

so for both choices of  $F$  in equation (45) what we need is  $\delta \hat{\rho}(t)/\delta w_j(s)$ , calculated through terms of order  $\sqrt{\gamma}$ . This can be calculated directly by integrating the differential equation of equation (40). In terms of the interaction picture operators

$$A_j(s-t) = e^{\frac{i}{\hbar}H(s-t)} A_j e^{-\frac{i}{\hbar}H(s-t)}, \quad (47)$$

a simple calculation gives

$$\frac{\delta \hat{\rho}(t)}{\delta w_j(s)} = \sqrt{\gamma}[i\xi_I[A_j(s-t), \hat{\rho}(t)] + \xi_R\{A_j(s-t) - \langle A_j \rangle_s, \hat{\rho}(t)\}]. \quad (48)$$

Substituting this into equation (44) gives a lengthy expression, which on algebraic simplification, and noting that  $\langle A_j(s-t) \rangle_t = \langle A_j \rangle_s$ , gives a result that may be summarized as follows. Let us abbreviate

$$S_{ij} \equiv \sum_{i,j=1}^N \int_0^t ds D_{ij}(t, s), \quad (49)$$

so that  $S_{ij}$  acting on a function of  $t, s$  gives a function only of  $t$ . Then we find

$$\begin{aligned} \frac{d\rho(t)}{dt} = & -\frac{i}{\hbar}[H, \rho(t)] + \gamma(\xi_R^2 + \xi_I^2)S_{ij}[A_i \rho(t) A_j(s-t) \\ & + A_j(s-t)\rho(t)A_i - A_i A_j(s-t)\rho(t) - \rho(t)A_j(s-t)A_i] \\ & + \gamma[S_{ij}(\xi_R^2 F_{ij} + i\xi_I \xi_R C_{ij}) + B_{\text{ASA}}, \rho(t)] \\ & + \gamma\{S_{ij}[\xi_R^2(C_{ij} - \langle C_{ij} \rangle_t) + i\xi_I \xi_R(F_{ij} - \langle F_{ij} \rangle_t)] + B_{\text{SA}} - \langle B_{\text{SA}} \rangle_t, \rho(t)\}, \end{aligned} \quad (50)$$

where we have introduced the condensed notations

$$\begin{aligned} F_{ij} &= [A_i, A_j(s-t)], \\ C_{ij} &= \{A_i, A_j(s-t)\} - 2A_i \langle A_j(s-t) \rangle_t - 2 \langle A_i \rangle_t A_j(s-t). \end{aligned} \quad (51)$$

Let us now make the following choice of the previously undetermined operators  $B_{SA}$  and  $B_{ASA}$ ,

$$B_{SA} = -S_{ij}(\xi_R^2 C_{ij} + i\xi_I \xi_R F_{ij}), \quad B_{ASA} = -S_{ij}(\xi_R^2 F_{ij} + i\xi_I \xi_R C_{ij}). \quad (52)$$

Then the final two lines of equation (50) cancel to zero, and we are left with

$$\begin{aligned} \frac{d\rho(t)}{dt} &= -\frac{i}{\hbar} [H, \rho(t)] + \gamma(\xi_R^2 + \xi_I^2) \sum_{i,j=1}^N \int_0^t ds D_{ij}(t,s) [A_i \rho(t) A_j(s-t) \\ &\quad + A_j(s-t) \rho(t) A_i - A_i A_j(s-t) \rho(t) - \rho(t) A_j(s-t) A_i]. \end{aligned} \quad (53)$$

Thus, we recover from this approach the evolution of equation (39) of section 2. In the Markovian limit in which  $D_{ij}(s, t)$  decays rapidly when  $s$  is not close to  $t$ , we can approximate  $s-t \simeq 0$  in equation (53). We then have  $A_j(s-t) \simeq A_j$ , which by assumption commutes with  $A_i$ , and the evolution of equation (53) takes the first standard form of a Lindblad evolution,

$$\begin{aligned} \frac{d\rho(t)}{dt} &= -\frac{i}{\hbar} [H, \rho(t)] + \gamma(\xi_R^2 + \xi_I^2) \sum_{i,j=1}^N \int_0^t ds D_{ij}(t,s) \\ &\quad \times [A_i \rho(t) A_j + A_j \rho(t) A_i - \{A_i A_j, \rho(t)\}]. \end{aligned} \quad (54)$$

Equation (54) also follows, without the Markovian approximation, if one neglects the Hamiltonian  $H$ , since then the interaction and Schrödinger pictures coincide, and one has exactly  $A_j(s-t) = A_j$ .

## 4. Change of measure and other concluding remarks

### 4.1. Change of measure

In sections 2 and 3, we have given two different derivations of the time evolution equation for the normalized wavefunction  $|\psi(t)\rangle$ , and of the corresponding evolution equation for the density matrix  $\rho(t)$ . By construction, the density matrix evolutions of equations (39) and (53) are the same. However, a comparison of the operators  $O_{SA}$  and  $O_{ASA}$  of section 2 with the operators  $B_{SA}$  and  $B_{ASA}$  of section 3 shows that these are not the same, and hence the corresponding evolution equations for  $|\psi(t)\rangle$  of equations (35) and (40) are not the same. This means that our two constructions correspond to inequivalent unravelings of the same density matrix evolution: in general the noises, expectations and wavefunctions  $|\psi(t)\rangle$  of sections 2 and 3 are not the same, even though they lead to the same evolution equation for the noise-averaged density matrix.

However, in certain special cases the formulations of sections 2 and 3 are related by a time-dependent shift in the mean values of the Gaussian noise variables. In order for the functions  $|\psi(t)\rangle$  used in the two derivations to be identical, they must obey the same time evolution equation. Changing notation, by denoting the noise of section 3 as  $\tilde{w}_i(t)$ , we find that the evolution equations for  $|\psi(t)\rangle$  of equation (35) and equation (40) become identical when the noises  $\tilde{w}_i(t)$  and  $w_i(t)$  are related to leading order in  $\sqrt{\gamma}$  by

$$\sum_{i=1}^N \tilde{w}_i(t) A_i = \sum_{i=1}^N w_i(t) A_i - 2\sqrt{\gamma} \xi_R \sum_{i,j=1}^N \int_0^t ds D_{ij}(t,s) [A_i \langle A_j(s-t) \rangle_t + \langle A_i \rangle_t A_j(s-t)], \quad (55)$$

with

$$\mathbb{E}_{\mathbb{R}}[\tilde{w}_i(t)\tilde{w}_j(s)] = \mathbb{E}_{\mathbb{Q}}[w_i(t)w_j(s)] = D_{ij}(t, s) \quad (56)$$

to zeroth order in  $\sqrt{\gamma}$ .

Equation (55) is consistent only if the operators  $A_j(s-t)$  can be expanded over a basis of the time zero operators  $A_i$ , with  $c$ -number coefficients, that is

$$A_j(s-t) = \sum_{i=1}^N K_{ji}(s-t)A_i. \quad (57)$$

This is automatically true (i) when the Hamiltonian  $H$  vanishes, since then  $A_j(s-t)$  is time independent, and (ii) in the white-noise limit, since then  $D_{ij}(t, s) \propto \delta(t-s)$ , and so equation (55) only involves  $A_j(0) = A_j$ . It is approximately true (iii) whenever a Markovian approximation to the time evolution is valid. In general, however, equation (55) involves on the right-hand side operators  $A_j(s-t)$  that are linearly independent of the operators  $A_i$ , and so cannot be satisfied.

When equation (57) holds, then equation (55) simplifies to take the form

$$\sum_{i=1}^N \tilde{w}_i(t)A_i = \sum_{i=1}^N w_i(t)A_i - \sum_{i=1}^N K_i(t)A_i, \quad (58)$$

with  $K_i(t)$  given by

$$K_i(t) = 2\sqrt{\gamma}\xi_{\mathbb{R}} \sum_{j=1}^N \int_0^t ds \left[ D_{ij}(t, s)\langle A_j(s-t) \rangle_t + \sum_{m=1}^N D_{jm}(s-t)\langle A_j \rangle_t K_m(s-t) \right]. \quad (59)$$

Equation (58) can clearly be satisfied by making a  $c$ -number shift in the noise variable for each  $i$ ,

$$\tilde{w}_i(t) = w_i(t) - K_i(t) \quad (60)$$

The relation at time  $t$  between the measure  $\mathbb{R}$  of section 3, in which  $\tilde{w}_i(t)$  has zero mean, and the measure  $\mathbb{Q}$  of section 2, in which  $w_i(t)$  has zero mean, is then to first order in  $\sqrt{\gamma}$ , for any argument  $O$ ,

$$\mathbb{E}_{\mathbb{R}}[O] = \mathbb{E}_{\mathbb{Q}}[W(t)O]. \quad (61)$$

Here the weighting factor  $W(t)$  is given by

$$W(t) = 1 + \sum_{i=1}^N \int_0^t ds C_i(t, s)w_i(s), \quad (62)$$

where  $C_i(t, s)$  obeys the integral equation

$$K_i(t) = \sum_{j=1}^N \int_0^t ds D_{ij}(t, s)C_j(t, s). \quad (63)$$

This choice of  $C_i(t, s)$  guarantees that when  $O$  is taken as  $\tilde{w}_i$  in equation (61), one finds that  $\mathbb{E}_{\mathbb{R}}[\tilde{w}_i] = 0$ , as needed. An analogous  $c$ -number shift of the noise variable enters in comparing equation (5) of section 1 with the white-noise limit of equation (18) in section 2.1.

#### 4.2. Final remarks

In the preceding sections, we have shown that the white-noise formalism is robust under a generalization to the physically more realistic assumption of non-white noise. Both the proof of state vector reduction, and the ‘imaginary time’ trick for calculating physical effects of the noise, carry over to the non-white-noise case, to leading quadratic order in the noise strength.

We wish here to elaborate on some implications of our calculations for models of the non-white noise. We recall that the noise autocorrelation  $D_{ij}(t_1, t_2)$  is defined as the expectation  $\mathbb{E}_{\mathbb{Q}}[w_i(t_1)w_j(t_2)]$ , and that the condition for state vector reduction is that the time integral

$$F_{ij}(t) = \int_0^t ds D_{ij}(t, s) \quad (64)$$

should be a positive definite matrix in the limit as  $t \rightarrow \infty$ .<sup>8</sup> Let us now assume time translation invariance, which implies that  $D_{ij}(t, s) = D_{ij}(t - s)$ , and investigate what the requirement for state vector reduction means in terms of the spectral decomposition of  $D_{ij}$ . Writing

$$D_{ij}(t - s) = \int_0^\infty d\omega \gamma_{ij}(\omega) \cos \omega(t - s) \quad (65)$$

we have

$$F_{ij}(t) = \int_0^\infty d\omega \gamma_{ij}(\omega) \frac{\sin \omega t}{\omega} = \int_0^\infty \frac{du}{u} \sin u \gamma_{ij}(u/t). \quad (66)$$

Thus, assuming that  $\gamma(\omega)$  is smooth in the neighborhood of  $\omega = 0$ , we find

$$\lim_{t \rightarrow \infty} F_{ij}(t) = \gamma_{ij}(0) \int_0^\infty \frac{du}{u} \sin u = \gamma_{ij}(0)\pi. \quad (67)$$

Hence, the reduction requirement is satisfied when  $\gamma_{ij}(0)$  is a positive definite matrix in  $i, j$ . In particular, the spectral weight  $\gamma_{ij}(\omega)$  can have a cutoff at a finite upper limit  $\omega = \omega_{\max}$ , without in any way affecting the reduction argument. The possibility of such an upper cutoff has been discussed in the review of Bassi and Ghirardi [1], and as noted by Adler and Ramazanoglu [22], is suggested on physical grounds by existing upper limits on noise-induced gamma-ray emission.

Let us next consider the rate of secular energy increase induced by the noise, taking advantage of the ‘imaginary-noise’ trick to write the noise term as an addition to the Hamiltonian. We consider the simple model describing a particle of mass  $m$  moving in one dimension, with a non-white-noise coupling to its coordinate  $x$ , with Hamiltonian

$$H = \frac{p^2}{2m} - C w_t x. \quad (68)$$

Because the noise term does not commute with  $p$ , the kinetic energy  $p^2/(2m)$  increases over time. We have for the expected rate of energy gain,

$$\frac{d}{dt} \mathbb{E} \left[ \frac{p^2}{2m} \right] = m^{-1} \mathbb{E} \left[ p \frac{dp}{dt} \right]. \quad (69)$$

From the Heisenberg equations of motion implied by equation (68) we find  $dx/dt = p$ ,  $dp/dt = C w_t$ , and so

$$p(t) = p(0) + C \int_0^t du w_u. \quad (70)$$

<sup>8</sup> More generally, as can be seen from equation (24), the reduction requirement is satisfied when  $\int_0^t ds F_{ij}(s) \rightarrow \infty$  as  $t \rightarrow \infty$ .

Hence the rate of energy gain is

$$\frac{d}{dt} \mathbb{E} \left[ \frac{p^2}{2m} \right] = \frac{C^2}{m} \int_0^t du \mathbb{E} [w_t w_u] = \frac{C^2}{m} \int_0^t du D(t, u) = \frac{C^2}{m} F(t), \quad (71)$$

and therefore is governed by the same integral over the autocorrelation function as the reduction rate. Thus, when the reduction condition  $\lim_{t \rightarrow \infty} F(t) > 0$  is obeyed, the rate of noise-induced energy production is nonzero at large times, a result which readily generalizes to more realistic reduction models. As reviewed in [3], this leads to various upper bounds on the noise strength. This conclusion can be evaded in the generic case of multiple operators  $A_i$ , if the only nonvanishing eigenvalue of  $F_{ij}(t)$  as  $t \rightarrow \infty$  is the one associated with the total energy. An interesting model where this case is realized, but in which localizing reduction still occurs in an approximate sense, is given by taking  $D_{ij}(t, s)$  to be a correlation function associated with thermal noise, as might be expected if state vector reduction is induced by some type of cosmological relic field. A detailed examination of the thermal noise model will be given elsewhere [23].

### Acknowledgments

The work of SLA was supported in part by the Department of Energy under grant no DE-FG02-90ER40542, and part of the work was done while he was at the Aspen Center for Physics. The work of AB was supported in part by DGF (Germany) and by the EU grant ERG 044941-STOCH-EQ; part of his work was done while visiting the Institute for Advanced Study in Princeton, which he wishes to thank for its support, and Clare Hall in Cambridge. We wish to thank Philip Pearle for helpful correspondence.

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